Equivalence of energy methods in stability theory

Petre Birtea, Mircea Puta

Abstract

We will prove the equivalence of three methods, the so called energy methods, in order to establish the stability of an equilibrium point for a dynamical system. We will illustrate by examples that this result simplifies enormously the amount of computations especially when the stability cannot be decided with one of the three methods.

MSC: 37C10, 37C75.
Keywords: dynamical systems, stability theory.

1 Introduction.

Let M be a smooth manifold and

$$\dot{x} = f(x) \tag{1.1}$$

be a dynamical system on M given by the vector field $f \in \mathfrak{X}(M)$ and suppose $x_e \in M$ is an equilibrium state for (1.1), i.e. $f(x_e) = 0$. The problem of nonlinear stability of equilibrium states is a very old one and the most know and remarkable results were obtained by Lyapunov [5]. They are based on finding what is called a Lyapunov function $V \in C^1(M, \mathbb{R})$ such that:

(i) $V(x_e) = 0$

(ii) $V(x) > 0$, for $x \neq x_e$

(iii) $\dot{V} \leq 0$, where $\dot{V}$ is the derivative of V along the trajectories of (1.1).

In practice it is sometimes very difficult to find such a function. In many situations one can use constants of motion as Lyapunov functions, i.e. functions $V : M \to \mathbb{R}$ such that $\dot{V} = 0$. This was extensively used in the context of Hamilton-Poisson systems where the Hamiltonian and the Casimirs of the Poisson structure are constants of motion. The methods for studying stability using constants of motion are the so called energy methods. The most general results using this methods for establishing stability can be found in [9] and [6]. Since in the present paper we are discussing local nonlinear stability we can replace, by considering a coordinate chart around the equilibrium x_e, the manifold M with $\mathbb{R}^n$, where n is the dimension of M.

In 1965 Arnold [1] gives the following criteria for determining nonlinear stability for an equilibrium point of (1.1).

Theorem 1.1 (The Arnold method [1]) *Let $C_1, \ldots, C_k \in C^2(\mathbb{R}^n, \mathbb{R})$ be constants of motion for the equation (1.1) and $F_i \in C^2(\mathbb{R}^n \times \mathbb{R}^{k-1}, \mathbb{R})$ be the smooth function given by:*

$$F_i(x, \lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_k) \stackrel{def}{:=} C_i(x) - \lambda_1 C_1(x) - \cdots - \widehat{\lambda_i C_i(x)} - \cdots - \lambda_k C_k(x)$$

where $\widehat{g}$ means that the term g is omitted. If there exist constants $\lambda_1^*, \ldots, \widehat{\lambda_i^*}, \ldots, \lambda_k^*$ in $\mathbb{R}$ such that

(i) $\nabla_x F_i(x_e, \lambda_1^*, \dots, \widehat{\lambda_i^*}, \dots, \lambda_k^*) = 0$

(ii) $\nabla_{xx}^2 F_i(x_e, \lambda_1^*, \dots, \widehat{\lambda_i^*}, \dots, \lambda_k^*)_{|W \times W}$ *is positive or negative definite, where*

$$W := \bigcap_{\substack{j=1 \\ j \neq i}}^{k} \ker dC_j(x_e),$$

then x_e is nonlinear stable.

Later, in 1985, Holm, Marsden, Ratiu and Weinstein [4] give another method for establishing stability of an Hamilton-Poisson system, the so called Energy-Casimir method.

Theorem 1.2 (The Energy-Casimir method [4]) *Let $C_1, C_2, \dots, C_k \in C^2(\mathbb{R}^n, \mathbb{R})$ be constants of motion for the equation (1.1). If there exist $\varphi_1, \dots, \widehat{\varphi}_i, \dots, \varphi_k \in C^2(\mathbb{R}, \mathbb{R})$ such that:*

(i) $\nabla_x (C_i + \varphi_1(C_1) + \dots + \widehat{\varphi_i(C_i)} + \dots + \varphi_k(C_k))(x_e) = 0$

(ii) $\nabla_{xx}^2 (C_i + \varphi_1(C_1) + \dots + \widehat{\varphi_i(C_i)} + \dots + \varphi_k(C_k))(x_e)$ *is positive or negative definite,*

then x_e is nonlinear stable.

The above result has also an infinite dimensional analogue for Hamilton-Poison systems on Banach spaces, see [4].

Studying the stability of relative equilibria, in 1998, Ortega and Ratiu [7] obtain, as a corollary of their results about stability of relative equilibria, the following theorem.

Theorem 1.3 (The Ortega-Ratiu method [7]) *Let $C_1, \dots, C_k \in C^2(\mathbb{R}^n, \mathbb{R})$ be constants of motion for the equation (1.1). If there exist $\varphi_1, \dots, \widehat{\varphi}_i, \dots, \varphi_k \in C^2(\mathbb{R}, \mathbb{R})$ such that:*

(i) $\nabla_x (C_i + \varphi_1(C_1) + \dots + \widehat{\varphi_i(C_i)} + \dots + \varphi_k(C_k))(x_e) = 0$

(ii) $\nabla_{xx}^2 (C_i + \varphi_1(C_1) + \dots + \widehat{\varphi_i(C_i)} + \dots + \varphi_k(C_k))(x_e)_{|\widetilde{W} \times \widetilde{W}}$ *is positive or negative definite, where*

$$\widetilde{W} := \bigcap_{\substack{j=1 \\ j \neq i}}^{k} \ker(d\varphi_j(C_j))(x_e),$$

then x_e is nonlinear stable.

The aim of our paper is to prove the equivalence of these three methods. This shows that when x_e is an equilibrium point for (1.1) and we choose $C_1, \dots, C_k$ as a set of constants of motion, if we conclude stability of x_e with one of the methods, then the other two will also give stability of x_e. Thus we can choose the most convenient method from the computational point of view. Since computations can become cumbersome in some examples it is important to know that if we cannot conclude stability of x_e using the set $C_1, \dots, C_k$ of constants of motion with one of the methods, then we cannot conclude stability of x_e by applying the other two methods using the same set $C_1, \dots, C_k$ of constants of motion.

2 Equivalence of the three methods

In order for the paper to be self-contained we will start by proving Arnold's result on stability since in his original paper [1] the proof was omitted. In order to do this we need the following preliminary results which will play a crucial role in all that follows.

We will begin by establishing the notations and conventions to be used throughout this paper. A vector $x \in \mathbb{R}^n$ will be considered as a column vector or a $n \times 1$ matrix. Its transpose will be a row vector or a $1 \times n$ matrix.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 real valued function. The gradient of f at a point $x \in \mathbb{R}^n$ is defined as the column vector

$$\nabla f(x) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1}(x) \\ \vdots \\ \dfrac{\partial f}{\partial x_n}(x) \end{bmatrix}.$$

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a vector valued map, then it will be represented as a column vector of its component functions $f_1, \ldots, f_m$, namely

$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix}.$$

If $f \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, then we introduce the notation

$$\nabla f(x) := \left[\nabla f_1(x) \ldots \nabla f_m(x)\right],$$

where $\nabla f(x)$ is a $n \times m$ matrix which has as columns the gradient vectors $\nabla f_1(x), \ldots, \nabla f_m(x)$. Note that the transpose matrix $\nabla f(x)^T$ is the Jacobian matrix of f at the point $x \in U$.

Let $f : \mathbb{R}^{n+k} \to \mathbb{R}$ be a C^2 real valued function and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^k$. We will use the following notations,

$$\nabla_x f(x,y) = \begin{bmatrix} \dfrac{\partial f(x,y)}{\partial x_1} \\ \vdots \\ \dfrac{\partial f(x,y)}{\partial x_n} \end{bmatrix}, \quad \nabla_y f(x,y) = \begin{bmatrix} \dfrac{\partial f(x,y)}{\partial y_1} \\ \vdots \\ \dfrac{\partial f(x,y)}{\partial y_k} \end{bmatrix},$$

$$\nabla^2_{xx} f(x,y) = \left[\frac{\partial^2 f(x,y)}{\partial x_i \partial x_j}\right], \quad \nabla^2_{xy} f(x,y) = \left[\frac{\partial^2 f(x,y)}{\partial x_i \partial y_j}\right],$$

$$\nabla^2_{yy} f(x,y) = \left[\frac{\partial^2 f(x,y)}{\partial y_i \partial y_j}\right].$$

For the proof of Theorem 1.1 we will need the following result that can be found in references [2] and [8] .

Proposition 2.1 *Let P be a symmetric $n \times n$ matrix and Q a positive semidefinite symmetric $n \times n$ matrix. We assume that*

$$x^T P x > 0,$$

for all $x \in \mathbb{R}^n$, $x \neq 0$ *satisfying* $x^T Q x = 0$. *Then there exists a scalar* $\alpha \in \mathbb{R}$ *such that*

$$P + \alpha Q > 0.$$

Proof. We will prove by contradiction. Then for every integer k, there exists a vector $x_k \in \mathbb{R}^n$ with $\|x_k\| = 1$ such that:

$$x_k^T P x_k + k x_k^T Q x_k \leq 0. \tag{2.1}$$

The sequence (x_k) is bounded and consequently it has a subsequence, that we will denote also by (x_k), converging to a vector $\overline{x} \in \mathbb{R}^n$ with $\|\overline{x}\| = 1$. Taking the limit in (2.1) we obtain

$$\overline{x}^T P \overline{x} + \lim_{k \to \infty} (k x_k^T Q x_k) \leq 0. \tag{2.2}$$

Since

$$x_k^T Q x_k \geq 0,$$

the inequality (2.2) implies that $(x_k^T Q x_k)$ converges to zero and hence $\overline{x}^T Q \overline{x} = 0$.

It follows from the hypothesis that $\overline{x}^T P \overline{x} > 0$ and this contradicts (2.2). ∎

Let x_e be an equilibrium point for the dynamic (1.1) and let $C_1, \ldots, C_k \in C^1(\mathbb{R}^m, \mathbb{R})$ be a set of constants of notion for the dynamic (1.1). We define the following quadratic form,

$$x^T Q_i x := \sum_{\substack{j=1 \\ j \neq i}}^{k} x^T \nabla C_j(x_e) (\nabla C_j(x_e))^T x. \tag{2.3}$$

We have the following characterization for the vector subspace W defined in Theorem 1.1.

Lemma 2.1

$$x^T Q_i x = 0 \Leftrightarrow x \in W = \bigcap_{\substack{j=1 \\ j \neq i}}^{k} \ker dC_j.$$

Proof.

$$\begin{aligned}
x^T Q_i x = 0 \quad &\Leftrightarrow \quad \sum_{\substack{j=1 \\ j \neq i}}^{k} x^T \nabla C_j(x_e)(\nabla C_j(x_e))^T x = 0 \\
&\Leftrightarrow \quad \sum_{\substack{j=1 \\ j \neq i}}^{k} ((\nabla C_j(x_e))^T x)((\nabla C_j(x_e))^T x) = 0 \\
&\Leftrightarrow \quad \sum_{\substack{j=1 \\ j \neq i}}^{k} ((\nabla C_j(x_e))^T x)^2 = 0 \\
&\Leftrightarrow \quad (\nabla C_j(x_e))^T x = 0, \quad \forall\ j = \overline{1,k}, j \neq i \\
&\Leftrightarrow \quad x \in W.
\end{aligned}$$

∎

Proof of Theorem 1.1. Let $L_{i,\alpha_i} \in C^2(\mathbb{R}^n \times \mathbb{R}^{k-1}, \mathbb{R})$ be the function defined by

$$\begin{aligned} L_{i,\alpha_i}(x, \lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_k) &:= F_i(x, \lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_k) \\ &+ \frac{\alpha_i}{2} \sum_{\substack{j=1 \\ j \neq i}}^{k} [C_j(x) - C_j(x_e)]^2, \end{aligned}$$

where $\alpha_i \in \mathbb{R}$ will be determined later.

A simple computation shows us that

$$\begin{aligned} \nabla_x L_{i,\alpha_i}(x, \lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_k) &= \nabla_x F_i(x, \lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_k) \\ &+ \alpha_i \sum_{\substack{j=1 \\ j \neq i}}^{k} (C_j(x) - C_j(x_e)) \nabla C_j(x) \end{aligned}$$

and

$$\begin{aligned} \nabla_{xx} L_{i,\alpha_i}(x_e, \lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_k) &= \nabla_{xx} F_i(x_e, \lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_k) \\ &+ \alpha_i \sum_{\substack{j=1 \\ j \neq i}}^{k} \nabla_x C_j(x_e) (\nabla C_j(x_e))^T \\ &= P_i + \alpha_i Q_i, \end{aligned}$$

where $Q_i = \sum\limits_{\substack{j=1 \\ j \neq i}}^{k} \nabla_x C_j(x_e)(\nabla C_j(x_e))^T$ is the $n \times n$ symmetric matrix defined by (2.3).

The hypothesis (i) implies that $\nabla_x L_{i,\alpha_i}(x_e, \lambda_1^*, \ldots, \widehat{\lambda_i^*}, \ldots, \lambda_k^*) = 0$. As a consequence of the hypothesis (ii) and Proposition 2.1 we can find $\alpha_i^* \in \mathbb{R}$ such that $P_i + \alpha_i^* Q_i > 0$ and thus $L_{i,\alpha_i^*}(x) > 0$ for $x \neq x_e$ in a small neighborhood of the equilibrium point x_e.

Let us define now the function $V_{i,\alpha_i^*} \in C^2(\mathbb{R}^n, \mathbb{R})$ by the following relation,

$$V_{i,\alpha_i^*}(x) = L_{i,\alpha_i^*}(x, \lambda_1^*, \ldots, \widehat{\lambda_i^*}, \ldots, \lambda_k^*) - L_{i,\alpha_i^*}(x_e, \lambda_1^*, \ldots, \widehat{\lambda_i^*}, \ldots, \lambda_k^*).$$

It is easy to see that V_{i,α_i^*} is a Lyapunov function and consequently via Lyapunov's theorem the equilibrium state x_e is nonlinear stable. ■

The proofs of Theorem 1.2 and Theorem 1.3 can be found in the original papers [4] and [7]. They are also based on finding a corresponding Lyapunov function.

Now we will prove the main result of this paper.

Theorem 2.2 *Let $C_1, \ldots, C_k \in C^2(\mathbb{R}^n, \mathbb{R})$ be a set of constants of motion for the dynamic (1.1). Then the following statements are equivalent:*

(a) hypotheses of Theorem 1.1 hold;

(b) hypotheses of Theorem 1.2 hold;

(c) hypotheses of Theorem 1.3 hold.

Each of the above statements implies nonlinear stability.

Proof. "$(a) \Rightarrow (b)$" Assume that the hypotheses of Theorem 1.1 hold. Consider the following functions $\varphi_j : \mathbb{R} \to \mathbb{R}$, $\varphi_j(t) = -\lambda_j^* t + \frac{\alpha_i}{2}(t - C_j(x_e))^2$, for $j \neq i$ and $\alpha_j \in \mathbb{R}$ arbitrary for the moment, and λ_j^* given in Theorem 1.1. As in the proof of Theorem 1.1, the conditions (i) and (ii) of Theorem 1.1 imply the conditions (i) and (ii) of Theorem 1.2 for a certain α_j^* given by Proposition 2.1.

"$(b) \Rightarrow (c)$" This is obvious since positive or negative definiteness on the whole space implies positive or negative definiteness on the subspace $\widetilde{W}$.

"$(c) \Rightarrow (a)$". Assume that the hypotheses of Theorem 1.3 hold. Let $\lambda_j^* = -\varphi_j'(C_j(x_e))$ for $j \neq i$. It is obvious that condition (i) of Theorem 1.3 implies condition (i) of Theorem 1.1. Also because some of λ_j^*'s might be zero we have the inclusion $W \subseteq \widetilde{W}$. Then

$$
\begin{aligned}
& z^T \left[\nabla_{xx}^2 (C_i + \varphi_1(C_1) + \cdots + \widehat{\varphi_i(C_i)} + \cdots + \varphi_k(C_k))(x_e)\right] y \\
= \; & z^T \nabla_{xx}^2 C_i(x_e) y + \sum_{\substack{l=1 \\ l \neq i}}^{k} z^T \left(\varphi_l'(C_l(x_e)) \left[\frac{\partial^2 C_l(x_e)}{\partial x_i \partial x_j}\right]\right) y \\
+ \; & \sum_{\substack{l=1 \\ l \neq i}}^{k} \sum_{s,p=1}^{n} \varphi_l''(C_l(x_e)) z_s y_p \frac{\partial C_l(x_e)}{\partial x^s} \frac{\partial C_l(x_e)}{\partial x^p} \\
= \; & z^T \nabla_{xx}^2 F_i(x_e, \lambda_1^*, \ldots, \widehat{\lambda_i^*}, \ldots, \lambda_k^*) y + \sum_{\substack{l=1 \\ l \neq i}}^{k} \sum_{s,p=1}^{n} \varphi_l''(C_l(x_e)) z_s y_p \frac{\partial C_l(x_e)}{\partial x^s} \frac{\partial C_l(x_e)}{\partial x^p},
\end{aligned}
$$

for any $z, y \in \mathbb{R}^n$.

If we take $z, y \in W$ the second summand will be zero and consequently condition (ii) of Theorem 1.3 implies condition (ii) of Theorem 1.1. ■

In all of the three methods the stability is decided when a certain matrix is positive or negative definite. Consequently, Arnold's method seems to be the most economical since it requires definiteness of a smaller matrix than the other two methods.

Next we will discus the situation in which condition (i) of Theorem 1.1 is not satisfied. Or equivalently, when the vectors $\nabla C_i(x_e)$, $i \in \overline{1,k}$ are linear independent. Consequently, in a small neighborhood U_{x_e} of x_e they generate an integrable distribution whose leaves are the level sets of the map $F := (C_1, \ldots, C_k) : \mathbb{R}^n \to \mathbb{R}^k$. Eventually after shrinking U_{x_e} all the points in U_{x_e} are regular points for F. There exists a diffeomorphism $\phi : U_{x_e} \to (F^{-1}(F(x_e)) \cap U_{x_e}) \times V_{F(x_e)}$, where $V_{F(x_e)}$ is a small neighborhood of $F(x_e)$ in $\mathbb{R}^k$. Because $(C_1, \ldots, C_k)$ are constants of motion for the dynamic (1.1) we obtain $\phi_* f = (Y, 0)$, where $Y \in \mathfrak{X}\left(F^{-1}(F(x_e)) \cap U_{x_e}\right)$. If (y, z) are coordinates induced by ϕ on $(F^{-1}(F(x_e)) \cap U_{x_e}) \times V_{F(x_e)}$ from a set of coordinates around x_e then the equations of motion corresponding to the vector field $\phi_* f$ are

$$
\begin{aligned}
\dot{y} &= Y(y, z) \\
\dot{z} &= 0.
\end{aligned}
\tag{2.4}
$$

Moreover, $\phi(x_e) = (y_e, 0)$ and y_e is an equilibrium point for the dynamic generated by the vector field Y. The above system can be regarded as a bifurcation problem with $z \in V_{F(x_e)}$ the bifurcation parameter. We have the following result.

Theorem 2.3 *If $(C_1, \ldots, C_k)$ are constants of motion for the dynamic (1.1) which are linear independent at the equilibrium point x_e, then x_e is stable for the dynamic (1.1) if the equilibrium point y_e is stable for the dynamic generated by the vector field Y and $(y_e, 0)$ is not a bifurcation point for (2.4).*

This result was used in [3] for the stability problem of Ishii's equation. Given the conditions of the above theorem it is enough to study the stability of a dynamical system that has fewer variables. Nevertheless, the problem is not free of difficulties since one has to find a set of adapted coordinates around x_e for the local fibration generated by the map F.

3 Examples

3.1 The free rigid body

Theorem 2.2 asserts that if stability is obtained with one of the methods, then it can be obtained with the other two as well. Indeed, let us consider the Euler momentum equations:

$$\begin{cases} \dot{m}_1 = \left(\dfrac{1}{I_3} - \dfrac{1}{I_2}\right) m_2 m_3 \\[2ex] \dot{m}_2 = \left(\dfrac{1}{I_1} - \dfrac{1}{I_3}\right) m_1 m_3 \\[2ex] m_3 = \left(\dfrac{1}{I_2} - \dfrac{1}{I_1}\right) m_1 m_2 \end{cases}$$

where $I_1 > I_2 > I_3 > 0$. Then $x_e = (M, 0, 0)$ is an equilibrium point and $C_1(m_1, m_2, m_3) = \dfrac{1}{2}\left(\dfrac{m_1^2}{I_1} + \dfrac{m_2^2}{I_2} + \dfrac{m_3^2}{I_3}\right)$, and $C_2(m_1, m_2, m_3) = \dfrac{1}{2}(m_1^2 + m_2^2 + m_3^2)$ are two constants of motion.

We study the stability of $x_e = (M, 0, 0)$, $M \neq 0$ by using Arnold's method. Let $F_1 = C_1 - \lambda C_2$, then $\nabla F_1(x_e) = 0$ iff $\lambda = \dfrac{1}{I_1}$. Also

$$\nabla^2_{xx} F_1 \left(x_e, \frac{1}{I_1}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{1}{I_2} - \dfrac{1}{I_1} & 0 \\ 0 & 0 & \dfrac{1}{I_3} - \dfrac{1}{I_1} \end{bmatrix}$$

and $W = Span\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$. It is easy to see that

$$\nabla^2_{xx} F_1 \left(x_e, \frac{1}{I_1}\right)_{|W \times W} > 0$$

. This shows that $x_e = (M, 0, 0)$, $M \neq 0$ is nonlinear stable.

Next, we will prove the same stability result using the other two methods. We begin with the Energy-Casimir method. Let $H_\varphi = C_1 + \varphi(C_2)$. The first variation is given by

$$\delta H_\varphi \;=\; \frac{m_1}{I_1}\delta m_1 + \frac{m_2}{I_2}\delta m_2 + \frac{m_3}{I_2}\delta m_3 + \varphi'(m_1\delta m_1 + m_2\delta m_2 + m_3\delta m_3).$$

Then $\delta H_\varphi(M,0,0)=0$ is equivalent with $\varphi'\left(\frac{1}{2}M^2\right)=-\frac{1}{I_1}$. Also

$$\begin{aligned}\delta^2 H_\varphi(M,0,0) &= \left(\frac{1}{I_2}-\frac{1}{I_1}\right)(\delta m_2)^2+\left(\frac{1}{I_3}-\frac{1}{I_1}\right)(\delta m_3)^2\\ &+ \varphi''\left(\frac{1}{2}M^2\right)M^2(\delta m_1)^2\end{aligned}$$

is positive definite iff $\varphi''\left(\frac{1}{2}M^2\right)>0$.

We can take $\varphi(t)=\left(t-\frac{1}{2}M^2\right)^2-\frac{1}{I_1}t$ and conclude that $x_e=(M,0,0)$, $M\neq 0$ is nonlinear stable.

For Ortega-Ratiu's method we can take the same constant of motion used for applying Arnold's method, i.e. $F_1=C_1-\frac{1}{I_1}C_2$.

3.2 Lorenz five component model

We will show in this example that if the stability of an equilibrium point cannot be decided with one of the three methods then it cannot be decided with the other two either. This is what Theorem 2.2 is predicting. It simplifies enormously the computations in the sense that if we do the computations using one of the methods and obtain that the stability cannot be decided, then it is useless to do the computations using the other two methods and the same set of constants of motion.

To illustrate this, we will take the example of Lorenz five component model. The equations are

$$\begin{cases}\dot{x}_1=-x_2x_3+bx_2x_5\\ \dot{x}_2=x_1x_3-bx_1x_5\\ \dot{x}_3=-x_1x_2\\ \dot{x}_4=-\dfrac{x_5}{\varepsilon}\\ \dot{x}_5=\dfrac{x_4}{\varepsilon}+bx_1x_5\end{cases}$$

where $b,\varepsilon\in\mathbb{R}^*$, $x_e=(0,0,M,0,0)$, $M\neq 0$ is an equilibrium point and $C_1(x_1,\dots,x_5)=\frac{1}{2}(x_1^2+2x_2^2+x_3^2+x_4^2+x_5^2)$, and $C_2=\frac{1}{2}(x_1^2+x_2^2)$ are constants of motion.

We try to apply Arnold's method. Take $F_1=C_1-\lambda C_2$. Then $\nabla F_1(x_e)=0$ is impossible for any $\lambda\in\mathbb{R}$. We have another possibility for choosing a constant of motion. Let $F_2=C_2-\lambda C_1$. Then $\nabla F_2(x_e)=0$ iff $\lambda=0$. Also

$$\nabla^2_{xx}F_2\left(x_e,0\right)=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

and

$$W = Span\left(\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right).$$

It is easy to see that $\nabla^2_{xx}F_2(x_e,0)_{|W\times W}$ is not definite.

Now we try to apply the Energy-Casimir method. Let $H^1_\varphi = C_1+\varphi(C_2)$. Then $\delta H^1_\varphi(x_e)=0$ is impossible for any $\varphi\in C^2(\mathbb{R},\mathbb{R})$. We take the other possibility, namely $H^2_\varphi = C_2+\varphi(C_1)$. Then we have

$$\delta H^2_\varphi = x_1\delta x_1 + x_2\delta x_2 + \varphi'(x_1\delta x_1 + 2x_2\delta x_2 + x_3\delta x_3 + x_4\delta x_4 + x_5\delta x_5).$$

Consequently $\delta H^2_\varphi(x_e)=0$ iff $\varphi'\left(\frac{1}{2}M^2\right)=0$. Also

$$\delta^2 H^2_\varphi = (\delta x_1)^2 + (\delta x_2)^2 + \varphi''\left(\frac{1}{2}M^2\right)(\delta x_3)^2$$

which is not definite.

Finally we will try to apply Ortega-Ratiu's method. Let $F = C_2+\varphi(C_1)$. We have that $\delta F(x_e)=0$ iff $\varphi'\left(\frac{1}{2}M^2\right)=0$ and then $\widetilde{W}=\mathbb{R}^5$. Also

$$\delta^2 F(x_e) = \begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & \varphi''\left(\frac{1}{2}M^2\right)M^2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

and consequently $\delta^2 F(x_e)_{|\widetilde{W}\times\widetilde{W}}$ is not definite for any choice of $\varphi\in C^2(\mathbb{R},\mathbb{R})$.

References

[1] **V. Arnold**; *Conditions for nonlinear stability of stationary plane curvilinear flows of an ideal fluid*, Doklady, tome 162, no 5 (1965), 773-777.

[2] **D. Bertsekas**; *Constrained optimization and Lagrange multiplier metods*, Academic Press, 1982.

[3] **P. Birtea, M. Puta** ; *On Ishii's equation* , C. R. Math. Acad. Sci. Paris, vol 341, no 2 (2005), 107-111.

[4] **D. Holm, J. Marsden, T. Ratiu, A. Weinstein**; *Nonlinear stability of fluid and plasma equilibria*, Physics Reports, vol 123, no 1 and 2 (1985), 1-116.

[5] **A.M. Lyapunov**; *Probléme Générale de la Stabilité du Mouvement*, Kharkov 1892. French translation in AA. Fac. Sci. Univ. Toulouse 9, 1907; reproduced in Ann. Math. Studies, 17, Princeton University Press 1949.

[6] **J.P. Ortega, V. Planas-Bielsa, T. Ratiu**; *Asymptotic and Lyapunov stability of constrained and Poisson equilibria*, J. Differential Equations, 214 (2005), 92-127.

[7] **J.P. Ortega, T. Ratiu**; *Nonlinear stability of singular relative periodic orbits in Hamiltonian systems with symmetry*, J. Geom. Phys., 32 (1999), 160-188.

[8] **G.W. Patrick**; *Relative equilibria in hamiltonian systems: the dynamic interpretation of nonlinear stability on a reduced phase space*, J. Geom. Phys., 9 (1992), 111-119.

[9] **G.W. Patrick, M. Roberts, C. Wulff**; *Stability of Poisson equilibria and Hamiltonian relative equilibria by energy methods*, Arch. Rational Mech. Anal., 174 (2004), 36–52.

P. Birtea
Departamentul de Matematică, Universitatea de Vest, RO–1900 Timişoara, Romania.
Département de Mathématiques de Besançon, Université de Franche-Comté, UFR des Sciences et Techniques, 16 route de Gray, F–25030 Besançon cédex, France.
Email: birtea@math.uvt.ro

M. Puta
Departamentul de Matematică, Universitatea de Vest, RO–1900 Timişoara, Romania.
Email: puta@math.uvt.ro